HW 4 (Additional).

1. Show the following "quasi-regularity" properties for outer-measure m^*: Let $m^*(A) < +\infty$. Then

(i) $m^*(A) = \inf\{ m(G) : \text{open } G \supseteq A\}$

(ii) $\exists$ a G_δ-set $H := \bigcap_{n \in \mathbb{N}} G_n \supseteq A$ s.t. $m(H) = m^*(A)$ (where each G_n is open).

2. Let I be an open interval with $0 < \ell(I) < \infty$ and let $z \in \mathbb{R}$ with $|z| < \frac{\ell(I)}{2}$. Show that $I \cap (I+z) \neq \phi$, and that $I \cup (I+z)$ is an interval of length $< \frac{3}{2}\ell(I)$.

3. Let $\{E_n : n \in \mathbb{N}\}$ be a sequence of measurable sets and let $E = \liminf E_n \left(:= \bigcup_{n=1}^{\infty} \bigcap_{k \geq n} E_k = \bigcup_{n=1}^{\infty} T_n \right.$ where $\left. T_n := \bigcap_{k \geq n} E_k \ \forall n \right)$. Show that

$$m(E) \leq \liminf_n m(E_n).$$

Hint: Justify each of the steps in the following

$$m(E) = \lim_n m(T_n) = \liminf_n m(T_n) \leq \liminf_n m(E_n).$$

4. Let $0 < m^*(A) < +\infty$, and let $\alpha \in (0,1)$. Show that $\exists$ an open interval I such that $m^*(A \cap I) > \alpha \cdot l(I)$.

Hint: By def. of m^*, $\exists$ COIC $\{I_n : n \in \mathbb{N}\}$ of A such that $\frac{1}{\alpha} m^*(A) > \sum_{n=1}^{\infty} l(I_n)$. Justify the following steps

$$\sum_{n=1}^{\infty} m^*(A \cap I_n) \geq m^*(A) > \sum_{n=1}^{\infty} \alpha \cdot l(I_n)$$

and so $\exists$ at least one term on LHS > the corresponding term on RHS.

5. Let $0 < m(E) < +\infty$, $\alpha \in (\frac{3}{4}, 1)$ and a nonempty open interval such that $m(E_0) > \alpha \cdot l(I)$, where $E_0 = E \cap I$ (see Q4). Show that $(-\delta, \delta) \subseteq E_0 - E_0 \subseteq E - E$ where $\delta = \frac{l(I)}{2}$; thus we have the following Steinhaus Theorem:

$0 < m(E) \Rightarrow E - E$ contains δ-neighborhood of 0

(the finiteness $m^*(A) < \infty$, & $m(E) < +\infty$ can be dropped in Q4, 5).

Hint. Suppose not: $\exists\, z$ with $|z| < \frac{l(I)}{2}$ s.t. $z \notin E_0 - E_0$. Then E_0 and $z + E_0$ must be disjoint and so, pl. justify

$$2\, m(E_0) = m(E_0) + m(z + E_0) = m\left(E_0 \cup (z + E_0)\right) \leq m\left(I \cup (z + I)\right) < \frac{3}{2} l(I),$$

by Q2. Thus $\frac{3}{4} l(I) < \alpha \cdot l(I) < m(E_0) < \frac{3}{4} l(I)$, a contradiction.

6. Each bounded closed subset $F \subseteq \mathbb{R}$ is compact in the sense that any open cover $\mathcal{C}$ of F has a finite subcover (Heine-Borel Theorem, cf. 2050).

7. Let $K \subseteq G \subseteq \mathbb{R}$ with compact K and open G. Then $\exists$ open set V containing 0 (the zero) such that $K+V \subseteq G$.

Hint. $\forall k \in K, \exists\, \delta_k > 0$ s.t. $k + V_{2\delta_k}(0) \subseteq G$. By Q6

$\exists\, k_1, k_2, \dots k_n \in K$ s.t. $K \subseteq \bigcup_{i=1}^{n} \left(k_i + V_{\delta_{k_i}}(0) \right)$.

Let $\delta = \min\{\delta_{k_1}, \delta_{k_2}, \dots \delta_{k_n}\}$. Then $K + V_\delta(0) \subseteq G$.

8.* (2nd proof of Steinhaus Th, cf Q5). Given $0 < m(E)$, we assume w.l.g (why?) that E is bounded. You can(?) use the outer & inner regularity with suitably small $\varepsilon > 0$ to find closed K and open G such that $K \subseteq E \subseteq G$ with $2 \cdot m(K) > m(G)$. By Q6, 7, K is compact and $K+V \subseteq G$ for some open set containing 0. Show that $V \subset K-K \; (\subseteq E-E)$, showing Steinhaus Th.

Hint: Let $v \in V$. Should $v+K$ be disjoint from K, one would

$$2\,m(K) = m(v+K) + m(K) = m\left((v+K) \cup_\circ K \right) \leq m(G),$$

contradicting our choice of K, G. Therefore $(v+K) \cap K$ is non-empty so $\exists\, k_1, k_2 \in K$ such that $v + k_1 = k_2$; hence $v \in K - K$.

9*. Let $0 < m(E) < +\infty$, $\alpha \in (\frac{1}{2}, 1)$ and I an open interval such that $m(E_0) > \alpha \cdot l(I)$, where $E_0 := E \cap I$. Find $\delta > 0$ s.t. $V_\delta(0) \subseteq E_0 - E_0$.